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Nontrivial independent sets of bipartite graphs and cross-intersecting families

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ABSTRACT

Let $G(X,Y)$ be a connected, non-complete bipartite graph with $|X| \leqslant |Y|$. An independent set A of $G(X,Y)$ is said to be trivial if $A \subseteq X$ or $A \subseteq Y$. Otherwise, A is nontrivial. By $\alpha(X,Y)$ we denote the maximum size of nontrivial independent sets of $G(X,Y)$. We prove that if the automorphism group of $G(X,Y)$ is transitive and primitive on X and Y, respectively, then $\alpha(X,Y) = |Y| - d(X) + 1$, where $d(X)$ is the degree of vertices in X. We also give the structures of maximum-sized nontrivial independent sets of $G(X,Y)$. Consequently, these results give the sizes and structures of maximum-sized cross-t-intersecting families of finite sets, finite vector spaces and permutations, as well as the sizes and structures of maximum-sized cross-Sperner families of finite sets and finite vector spaces.

1. Introduction

Let X be a finite set and, for $0 \leqslant k \leqslant |X|$, let $\binom{X}{k}$ denote the family of all k-subsets of X, and let S_X and A_X denote the symmetric group and alternating group on X, respectively. In particular, for positive integer n, let $[n]$ denote the set $\{1, 2, \dots, n\}$, $[k,n] = \{k+1, \dots, n\}$ for $k \leqslant n$, and abbreviate the symmetric group and alternating group on $[n]$ as S_n and A_n, respectively.

A family $\mathcal{A}$ of sets is said to be *t-intersecting* if $|A \cap B| \geqslant t$ holds for all $A, B \in \mathcal{A}$. Usually, $\mathcal{A}$ is called intersecting if $t = 1$. The celebrated Erdős–Ko–Rado theorem [10], says that if $\mathcal{A}$ is a t-intersecting family in $\binom{[n]}{k}$, then

$$|\mathcal{A}| \leqslant \binom{n-t}{k-t}$$

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for $n \geqslant n_0(k,t)$. The smallest $n_0(k,t) = (k-t+1)(t+1)$ was determined by Frankl [11] for $t \geqslant 15$ and subsequently determined by Wilson [30] for all t.

The Erdős–Ko–Rado theorem has many generalizations, analogues and variations. First, the finite sets were analogous to finite vector spaces, permutations and other mathematical objects. Second, intersecting families were generalized to cross-intersecting families: $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ are said to be *cross-t-intersecting* if $|A \cap B| \geqslant t$ for all $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_j$, $i \neq j$. Some typical results are listed as follows.

Let $\mathbb{F}_q$ be a finite field of order q, $V = V_n(q)$ an n-dimensional vector space over $\mathbb{F}_q$, and $\left[{V \atop k}\right]$ the set of all k-dimensional subspaces (or k-subspaces, for short) of V. Then the cardinality of $\left[{V \atop k}\right]$ equals $\left[{n \atop k}\right]_q = \prod_{i=0}^{k-1} \frac{q^{n-i}-1}{q^{k-i}-1}$. For brevity, we write $\left[{n \atop k}\right]$ rather than $\left[{n \atop k}\right]_q$. A subset $\mathcal{A}$ of $\left[{V \atop k}\right]$ is said to be a *t-intersecting* family if $\dim(A \cap B) \geqslant t$ for any $A, B \in \mathcal{A}$. The Erdős–Ko–Rado theorem for finite vector spaces says that if $\mathcal{A}$ is a t-intersecting family in $\left[{V \atop k}\right]$, then

$$|\mathcal{A}| \leqslant \max\left\{ \begin{bmatrix} n-t \\ k-t \end{bmatrix}, \begin{bmatrix} 2k-t \\ k \end{bmatrix} \right\}$$

for $n \geqslant 2k - t$. This theorem was first established by Hsieh [19] for $t = 1$ and $k < n/2$, then by Greene and Kleitman [15] for $t = 1$ and $k|n$, and finally by Frankl and Wilson [13] for the general case.

A subset A of S_n is said to be a *t-intersecting* family if any two permutations in A agree in at least t points, i.e. for any $\sigma, \tau \in A$, $|\{i \in [n]\colon\ \sigma(i) = \tau(i)\}| \geqslant t$. Deza and Frankl [8] showed that an intersecting family in S_n has size at most $(n-1)!$ and conjectured that for t fixed, and n sufficiently large depending on t, a t-intersecting family in S_n has size at most $(n-t)!$. Cameron and Ku [6] proved that an intersecting family of size $(n-1)!$ is a coset of the stabilizer of a point. A few alternative proofs of Cameron and Ku's result were given in [23,16,28]. Ku and Leader [22] also generalized this result to partial permutations (see also [24]). Ellis, Friedgut and Pilpel [9] proved Deza and Frankl's conjecture on t-intersecting families in S_n.

Hilton [17] investigated the cross-intersecting families in $\binom{[n]}{k}$: Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ be cross-intersecting families in $\binom{[n]}{k}$. If $k \leqslant n/2$, then

$$\sum_{i=1}^{m} |\mathcal{A}_i| \leqslant \begin{cases} \binom{n}{k} & \text{if } m \leqslant \frac{n}{k}; \\ m\binom{n-1}{k-1} & \text{if } m \geqslant \frac{n}{k}. \end{cases} \tag{1.1}$$

He also determined the structures of $\mathcal{A}_i$'s when equality holds. Borg [3] gives a simple proof of this theorem, and generalizes it to labeled sets [2], signed sets [5] and permutations [4]. We generalized this theorem to general symmetric systems [27], which comprise finite sets, finite vector spaces and permutations, etc.

Hilton and Milner [18] and Frankl and Tohushige [12] also investigated the sizes of a pair of cross-intersecting families: If $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ are cross-intersecting families with $n \geqslant a+b$, $a \leqslant b$, then $|\mathcal{A}| + |\mathcal{B}| \leqslant \binom{n}{b} - \binom{n-a}{b} + 1$.

A similar result was established for cross-Sperner families. Here, two families $\mathcal{A}$ and $\mathcal{B}$ are said to be *cross-Sperner* if there exist no $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with $A \subseteq B$ or $B \subseteq A$. Recently, Gerbner, Lemons, Palmer, Patkós and Szécsi [14] proved that if $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ are cross-Sperner, then $|\mathcal{A}| + |\mathcal{B}| \leqslant 2^n - 2^{\lfloor n/2 \rfloor} - 2^{\lceil n/2 \rceil} + 2$. They also determined the structure of $\mathcal{A}$ and $\mathcal{B}$ when equality holds.

The last two results actually afford an upper bound on the sizes of nontrivial independent sets in a bipartite graph.

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, define the neighborhood $N_G(v) = \{u \in V(G)\colon\ uv \in E(G)\}$, and for $A = \{v_1, \ldots, v_k\} \subseteq V(G)$ write $N_G(A) = N_G(v_1, \ldots, v_k) = \bigcup_{i=1}^{k} N_G(v_i)$. If there is no possibility of confusion, then we abbreviate $N_G(A)$ as $N(A)$. A subset A of $V(G)$ is an independent set of G if $A \cap N(A) = \emptyset$. A graph G is bipartite if $V(G)$ can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y. In this case, we denote the bipartite graph by $G(X,Y)$. An independent set A of $G(X,Y)$ is said to be *trivial* if $A \subseteq X$ or $A \subseteq Y$. In any other case, A is *nontrivial*. If every vertex in X is adjacent to every vertex in Y, then $G(X,Y)$ is called a complete bipartite graph. Clearly, a complete bipartite

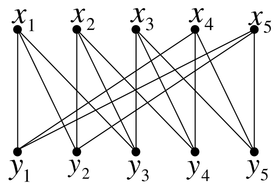

Fig. 1.

graph has only trivial independent sets. A bipartite graph $G(X, Y)$ is said to be *part-transitive* if there is a group Γ transitively acting on X and Y, respectively, and preserving the adjacency relation of the graph. Clearly, if $G(X, Y)$ is part-transitive, then every vertex of X (Y) has the same degree, written as $d(X)$ ($d(Y)$). By $\alpha(X, Y)$ and $I(X, Y)$ we denote the size and the set of maximum-sized nontrivial independent sets of $G(X, Y)$, respectively.

This paper elucidates $\alpha(X, Y)$ and $I(X, Y)$ for part-transitive bipartite graphs $G(X, Y)$. To this end, let us make a simple observation, as follows.

Let $G(X, Y)$ be a non-complete bipartite graph and let $A \cup B$ be a nontrivial independent set of $G(X, Y)$, where $A \subset X$ and $B \subset Y$. Then $A \subseteq X \backslash N(B)$ and $B \subseteq Y \backslash N(A)$, which implies that

$$|A| + |B| \leqslant \max\{|A| + |Y| - |N(A)|, |B| + |X| - |N(B)|\}.$$

From this one sees that

$$\alpha(X, Y) = \max\{|Y| - \epsilon(X), |X| - \epsilon(Y)\}, \tag{1.2}$$

where

$$\epsilon(X) = \min\{|N(A)| - |A|\colon\ A \subset X \text{ and } N(A) \neq Y\}$$

and

$$\epsilon(Y) = \min\{|N(B)| - |B|\colon\ B \subset Y \text{ and } N(B) \neq X\}.$$

A subset A of X is called a *fragment* of $G(X, Y)$ in X if $N(A) \neq Y$ and $|N(A)| - |A| = \epsilon(X)$. By $\mathcal{F}(X)$ we denote the set of all fragments in X. Similarly, we may define $\mathcal{F}(Y)$. Write $\mathcal{F}(X, Y) = \mathcal{F}(X) \cup \mathcal{F}(Y)$. An element $A \in \mathcal{F}(X, Y)$ is also called a *k-fragment* if $|A| = k$. As we shall see (Lemma 2.1), $|Y| - \epsilon(X) = |X| - \epsilon(Y)$, from which it follows that a maximum-sized nontrivial independent set is of the form $A \cup Y \backslash N(A)$ for $A \in \mathcal{F}(X)$. Therefore, in order to address our problem it suffices to determine $\mathcal{F}(X)$.

Note that most bipartite graphs presently considered have only 1-fragments, but there are actually bipartite graphs with sufficiently large nontrivial fragments. For example, let n and r be fixed positive integer with $r < n$, $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Define $x_i y_j$ to be an edge of $G(X, Y)$ if and only if $j \in \{i, i+1, \ldots, i+r-1\} \pmod n$ (see Fig. 1 for $n = 5$ and $r = 3$). It is easy to verify that $\{x_i, x_{i+1}, \ldots, x_{i+j}\}$ is a fragment in X, where $1 \leqslant j \leqslant n - r - 1$ and the subscripts are computed modulo n.

Let X be a finite set and let Γ be a group transitively acting on X. We say that the action of Γ on X is *primitive*, or Γ is primitive on X, if Γ preserves no nontrivial partition of X. In any other case, the action of Γ is *imprimitive*. It is easy to see that if the action of Γ on X is transitive and imprimitive, then there is a subset B of X such that $1 < |B| < |X|$ and $\gamma(B) \cap B = B$ or $\emptyset$ for every $\gamma \in G$. In this case, B is called an *imprimitive set* in X. It is well known that the action of Γ is primitive if and only if for each $a \in X$, the stabilizer of a, written as Γ_a defined to be the set $\{\gamma \in \Gamma\colon\ \gamma(a) = a\}$, is a maximal subgroup of Γ (cf. [20, Theorem 1.12]). Furthermore, a subset B of X is said to be *semi-imprimitive* if $1 < |B| < |X|$ and $|\gamma(B) \cap B| = 0, 1$ or $|B|$ for each $\gamma \in \Gamma$. Clearly, every 2-subset of X is semi-imprimitive.

The following are main results of this paper.

Theorem 1.1. *Let $G(X,Y)$ be a non-complete bipartite graph with $|X| \leqslant |Y|$. If $G(X,Y)$ is part-transitive and every fragment of $G(X,Y)$ is primitive under the action of a group Γ. Then $\alpha(X,Y) = |Y| - d(X) + 1$. Moreover,*

(1) *if $|X| < |Y|$, then X has only 1-fragments*;
(2) *if $|X| = |Y|$, then each fragment in X has size 1 or $|X| - d(X)$ unless there is a semi-imprimitive fragment in X or Y.*

As consequences of this theorem, we give the upper bounds of sizes of a pair of cross-t-intersecting families of finite sets, finite vector spaces and symmetric groups, as well as the upper bounds of sizes of a pair of cross-Sperner families of finite sets and finite vector spaces.

Theorem 1.2. *Let n, a, b, t be positive integers with $n \geqslant 4$, $a, b \geqslant 2$, $t < \min\{a, b\}$, $a + b < n + t$, $(n, t) \neq (a + b, 1)$ and $\binom{n}{a} \leqslant \binom{n}{b}$. If $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ are cross-t-intersecting, then*

$$|\mathcal{A}| + |\mathcal{B}| \leqslant \binom{n}{b} - \sum_{i=0}^{t-1} \binom{a}{i}\binom{n-a}{b-i} + 1. \tag{1.3}$$

Moreover equality holds if and only if one of the following holds:

(1) $\binom{n}{a} \leqslant \binom{n}{b}$ *and $\mathcal{A} = \{A\}$ and $\mathcal{B} = \{B \in \binom{[n]}{b}\colon |B \cap A| \geqslant t\}$ for some $A \in \binom{[n]}{a}$*;
(2) $\binom{n}{a} = \binom{n}{b}$ *and $\mathcal{B} = \{B\}$ and $\mathcal{A} = \{A \in \binom{[n]}{a}\colon |A \cap B| \geqslant t\}$ for some $B \in \binom{[n]}{b}$*;
(3) $(a, b, t) = (2, 2, 1)$ *and $\mathcal{A} = \mathcal{B} = \{C \in \binom{[n]}{2}\colon i \in C\}$ for some $i \in [n]$*;
(4) $(a, b, t) = (n-2, n-2, n-3)$ *and $\mathcal{A} = \mathcal{B} = \binom{A}{n-2}$ for some $A \in \binom{[n]}{n-1}$.*

Theorem 1.3. *Let n, a, b be positive integers with $a < b < n$. If $\mathcal{A} \subseteq \binom{[n]}{a}$ and $\mathcal{B} \subseteq \binom{[n]}{b}$ are cross-Sperner, then*

$$|\mathcal{A}| + |\mathcal{B}| \leqslant \max\left\{\binom{n}{b} - \binom{n-a}{b-a} + 1, \binom{n}{a} - \binom{b}{a} + 1\right\}. \tag{1.4}$$

Moreover equality holds if and only if one of the following holds:

(1) $\binom{n}{a} \leqslant \binom{n}{b}$ *and $\mathcal{A} = \{A\}$ and $\mathcal{B} = \{B \in \binom{[n]}{b}\colon B \not\supset A\}$ for some $A \in \binom{[n]}{a}$*;
(2) $\binom{n}{a} \geqslant \binom{n}{b}$ *and $\mathcal{B} = \{B\}$ and $\mathcal{A} = \{A \in \binom{[n]}{a}\colon A \not\subset B\}$ for some $B \in \binom{[n]}{b}$*;
(3) $(a, b) = (2, n-2)$ *and $\mathcal{A} = \{A \in \binom{[n]}{2}\colon i \in A\}$, $\mathcal{B} = \{B \in \binom{[n]}{n-2}\colon i \notin B\}$ for some $i \in [n]$.*

Theorem 1.4. *Let V be an n-dimensional vector space over the field of order q and let n, a, b, t be positive integers with $n \geqslant 4$, $a, b \geqslant 2$, $t < \min\{a, b\}$, $a + b < n + t$, and $\genfrac{[}{]}{0pt}{}{n}{a} \leqslant \genfrac{[}{]}{0pt}{}{n}{b}$. If $\mathcal{A} \subset \genfrac{[}{]}{0pt}{}{V}{a}$ and $\mathcal{B} \subset \genfrac{[}{]}{0pt}{}{V}{b}$ are cross-t-intersecting, then*

$$|\mathcal{A}| + |\mathcal{B}| \leqslant \begin{bmatrix} n \\ b \end{bmatrix} - \sum_{i=0}^{t-1} q^{(a-i)(b-i)} \begin{bmatrix} a \\ i \end{bmatrix}\begin{bmatrix} n-a \\ b-i \end{bmatrix} + 1. \tag{1.5}$$

Moreover equality holds if and only if one of the following holds:

(1) *$\mathcal{A} = \{A\}$ and $\mathcal{B} = \{B \in \genfrac{[}{]}{0pt}{}{V}{b}\colon \dim(A \cap B) \geqslant t\}$ for some $A \in \genfrac{[}{]}{0pt}{}{V}{a}$*;
(2) $\genfrac{[}{]}{0pt}{}{n}{a} = \genfrac{[}{]}{0pt}{}{n}{b}$ *and $\mathcal{A} = \{A \in \genfrac{[}{]}{0pt}{}{V}{b}\colon \dim(A \cap B) \geqslant t\}$ and $\mathcal{B} = \{B\}$ for some $B \in \genfrac{[}{]}{0pt}{}{V}{b}$.*

Theorem 1.5. *Let n, a, b be positive integers with $a < b < n$. If $\mathcal{A} \subseteq \genfrac{[}{]}{0pt}{}{V}{a}$ and $\mathcal{B} \subseteq \genfrac{[}{]}{0pt}{}{V}{b}$ are cross-Sperner, then*

$$|\mathcal{A}| + |\mathcal{B}| \leqslant \max\left\{\begin{bmatrix} n \\ b \end{bmatrix} - \begin{bmatrix} n-a \\ b-a \end{bmatrix} + 1, \begin{bmatrix} n \\ a \end{bmatrix} - \begin{bmatrix} b \\ a \end{bmatrix} + 1\right\}. \tag{1.6}$$

Moreover the equality holds if and only if one of the following holds:

(1) $\begin{bmatrix} n \\ a \end{bmatrix} \leqslant \begin{bmatrix} n \\ b \end{bmatrix}$ *and* $\mathcal{A} = \{A\}$ *and* $\mathcal{B} = \{B \in \begin{bmatrix} V \\ b \end{bmatrix}\colon\ B \not\supset A\}$ *for some* $A \in \begin{bmatrix} V \\ a \end{bmatrix}$;
(2) $\begin{bmatrix} n \\ a \end{bmatrix} \geqslant \begin{bmatrix} n \\ b \end{bmatrix}$ *and* $\mathcal{B} = \{B\}$ *and* $\mathcal{A} = \{A \in \begin{bmatrix} V \\ a \end{bmatrix}\colon\ A \not\subset B\}$ *for some* $B \in \begin{bmatrix} V \\ b \end{bmatrix}$.

Theorem 1.6. *Let n and t be positive integers with $n \geqslant 4$ and $t \leqslant n-2$. If $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting families in S_n, then*

$$|\mathcal{A}| + |\mathcal{B}| \leqslant n! - \sum_{i=0}^{t-1} \binom{n}{i} d_{n-i} + 1, \tag{1.7}$$

where d_{n-i} is the number of derangements in S_{n-i}. Moreover, equality holds if and only if $\{\mathcal{A}, \mathcal{B}\} = \{\{\sigma\}, \{\tau \in S_n\colon \sigma \text{ and } \tau \text{ are } t\text{-intersecting}\}\}$ for some $\sigma \in S_n$.

We shall investigate the nontrivial independent sets of bipartite graphs and complete the proof of Theorem 1.1 in the next section, and proved Theorem 1.2 in Section 3, Theorem 1.3 in Section 4, Theorem 1.4 in Section 5, Theorem 1.5 in Section 6, and Theorem 1.6 in Section 7.

2. Nontrivial independent sets of bipartite graphs

Before we start the proof of Theorem 1.1, we present two lemmas.

Lemma 2.1. *Let $G(X,Y)$ be a non-complete bipartite graph. Then, $|Y| - \epsilon(X) = |X| - \epsilon(Y)$, and*

(i) *$A \in \mathcal{F}(X)$ if and only if $Y \backslash N(A) \in \mathcal{F}(Y)$, and $N(Y \backslash N(A)) = X \backslash A$;*
(ii) *$A \cap B$ and $A \cup B$ are both in $\mathcal{F}(X)$ if $A, B \in \mathcal{F}(X)$, $A \cap B \neq \emptyset$ and $N(A \cup B) \neq Y$.*

Proof. Suppose $A \in \mathcal{F}(X)$ and put $C = Y \backslash N(A)$. Clearly, $N(C) \subseteq X \backslash A$. If $N(C) \neq X \backslash A$, writing $A' = X \backslash N(C)$, then $A \subsetneq A'$ and $N(A') = N(A)$. So $|N(A')| - |A'| < |N(A)| - |A| = \epsilon(X)$, yielding a contradiction. Hence $N(C) = X \backslash A$, and $|N(C)| - |C| = (|X| - |A|) - (|Y| - |N(A)|) = \epsilon(X) - |Y| + |X| \geqslant \epsilon(Y)$. Symmetrically, for $D \in \mathcal{F}(Y)$, putting $A = X \backslash N(D)$, we have $N(A) = Y \backslash D$ and $|N(A)| - |A| = (|Y| - |D|) - (|X| - |N(D)|) = \epsilon(Y) - |X| + |Y| \geqslant \epsilon(X)$. We then obtain that $\epsilon(X) + |X| = \epsilon(Y) + |Y|$, and (i) holds.

Now, suppose that $A, B \in \mathcal{F}(X)$, $A \cap B \neq \emptyset$ and $N(A \cup B) \neq Y$. Then $|N(A \cup B)| - |A \cup B| \geqslant \epsilon(X)$ and $|N(A \cap B)| - |A \cap B| \geqslant \epsilon(X)$. Note that $N(A \cup B) = N(A) \cup N(B)$ and $N(A \cap B) \subseteq N(A) \cap N(B)$. We have

$$\begin{aligned}
\epsilon(X) &\leqslant \bigl|N(A \cup B)\bigr| - |A \cup B| \\
&= \bigl|N(A)\bigr| + \bigl|N(B)\bigr| - \bigl|N(A) \cap N(B)\bigr| - |A| - |B| + |A \cap B| \\
&\leqslant 2\epsilon(X) - \bigl(\bigl|N(A \cap B)\bigr| - |A \cap B|\bigr) \leqslant \epsilon(X),
\end{aligned}$$

which implies that $|N(A \cup B)| - |A \cup B| = \epsilon(X)$ and $|N(A \cap B)| - |A \cap B| = \epsilon(X)$, hence (ii) holds. $\square$

From the first statement of this lemma it follows that there is a one to one correspondence $\phi : \mathcal{F}(X,Y) \mapsto \mathcal{F}(X,Y)$, where

$$\phi(A) = \begin{cases} Y \backslash N(A) & \text{if } A \in \mathcal{F}(X), \\ X \backslash N(A) & \text{if } A \in \mathcal{F}(Y). \end{cases}$$

Moreover, ϕ is an involution, i.e., $\phi^{-1} = \phi$, and $|A| + |\phi(A)| = \alpha(X,Y)$. A fragment is called *balanced* if $|A| = |\phi(A)|$. Clearly, all balanced fragments have the identical size $\frac{1}{2}\alpha(X,Y)$.

Lemma 2.2. *Let $G(X,Y)$ be a non-complete and part-transitive bipartite graph under the action of a group Γ. Suppose that $A \in \mathcal{F}(X,Y)$ such that $\emptyset \neq \gamma(A) \cap A \neq A$ for some $\gamma \in \Gamma$. If $|A| \leqslant |\phi(A)|$, then $A \cup \gamma(A)$ and $A \cap \gamma(A)$ are both in $\mathcal{F}(X,Y)$.*

Proof. Without loss of generality, suppose $A \in \mathcal{F}(X)$ and $|A| \leqslant |\phi(A)| = |Y \backslash N(A)|$. Because $|N(A)| = |A| + \epsilon(X)$ and $|N(A \cap \gamma(A))| \geqslant |A \cap \gamma(A)| + \epsilon(X)$,

$$\begin{aligned} \bigl|N\bigl(A \cup \gamma(A)\bigr)\bigr| &= 2\bigl|N(A)\bigr| - \bigl|N(A) \cap N\bigl(\gamma(A)\bigr)\bigr| \\ &\leqslant 2\bigl|N(A)\bigr| - \bigl|N\bigl(A \cap \gamma(A)\bigr)\bigr| \\ &\leqslant \bigl|N(A)\bigr| + |A| + \epsilon(X) - \bigl(\bigl|A \cap \gamma(A)\bigr| + \epsilon(X)\bigr) \\ &= \bigl|N(A)\bigr| + \bigl|A \backslash \gamma(A)\bigr| < \bigl|N(A)\bigr| + \bigl|Y \backslash N(A)\bigr| = |Y|. \end{aligned}$$

Then, by Lemma 2.1(ii), $A \cap \gamma(A)$ and $A \cup \gamma(A)$ are both in $\mathcal{F}(X)$. $\square$

From the above lemma it follows that if every element of $\mathcal{F}(X)$ ($\mathcal{F}(Y)$) is primitive and there is an $A \in \mathcal{F}(X)$ ($\mathcal{F}(Y)$) with $|A| \leqslant |\phi(A)|$, then $\mathcal{F}(X)$ ($\mathcal{F}(Y)$) contains a singleton. In particular, when $|X| = |Y|$ there are always two kinds of fragments in X: one is $\{a\}$ for $a \in X$, the other is $X \backslash N(b)$ for $b \in Y$. The former is a minimum-sized fragment, and the latter is maximum-sized one. We call the fragments of this kinds *trivial*. All others are *nontrivial*.

Proof of Theorem 1.1. From the above discussion we have that $\mathcal{F}(X,Y)$ contains a singleton, that is, $\alpha(X,Y) = \max\{|Y| - d(X) + 1, |X| - d(Y) + 1\}$. By counting the edges of $G(X,Y)$ we have $d(X)|X| = d(Y)|Y|$, so $d(X) = d(Y)|Y|/|X| \geqslant d(Y)$ because $|Y| \geqslant |X|$. Then

$$|Y| - |X| = d(X)|X|/d(Y) - |X| = \bigl(d(X) - d(Y)\bigr)|X|/d(Y) \geqslant d(X) - d(Y).$$

Equality holds if and only if $d(X) = d(Y)$ hence $|X| = |Y|$ because $|X| > d(Y)$. This proves that $|X| - d(Y) + 1 \leqslant |Y| - d(X) + 1$ and equality holds if and only if $|X| = |Y|$. In either cases, $\alpha(X,Y) = |Y| - d(X) + 1$.

We complete the proof via considering of two cases.

Case 1: $|X| < |Y|$. In this case we have seen that $\mathcal{F}(X)$ contains singletons while $\mathcal{F}(Y)$ does not. Now, let A be a maximum-sized element of $\mathcal{F}(X)$ and write $B = \phi(A) = Y \backslash N(A)$. Then B is a minimum-sized element of $\mathcal{F}(Y)$ with $|B| > 1$ and $\phi(B) = A$. Suppose $|A| > 1$. Since A and B are primitive, there are $\sigma, \gamma \in \Gamma$ such that $\sigma(A) \cap A \neq \emptyset$, $\sigma(A) \neq A$, $\gamma(B) \cap B \neq \emptyset$ and $\gamma(B) \neq B$. From this and Lemma 2.2 it follows that if $|A| \leqslant |B| = |\phi(A)|$, then $\sigma(A) \cup A \in \mathcal{F}(X)$, contradicting the maximality of $|A|$; if $|B| \leqslant |A| = |\phi(B)|$, then $\gamma(B) \cap B \in \mathcal{F}(Y)$, contradicting the minimality of $|B|$. This proves that $|A| = 1$ for every $A \in \mathcal{F}(X)$.

Case 2: $|X| = |Y|$. In this case, if there is a nontrivial fragment in X or in Y, let A be a minimum-sized one. Then $1 < |A| \leqslant |\phi(A)|$. From Lemma 2.2 it follows that for every $\gamma \in \Gamma$, $\gamma(A) \cap A$ is a fragment whenever $\gamma(A) \cap A \neq \emptyset$. Then, the minimality of $|A|$ implies that $|\gamma(A) \cap A| = 0$, 1 or $|A|$, for every $\gamma \in \Gamma$, i.e., A is semi-imprimitive. $\square$

For applications of the theorem, we further elucidate the fragments in the remainder of this section.

However, as we shall see, whether or not a bipartite graph has sufficiently large fragments depends if it has a 2-fragment.

Proposition 2.3. *Let $G(X,Y)$ be a non-complete bipartite graph with $|X| = |Y|$ and $\epsilon(X) = d(X) - 1$, and let Γ be a group part-transitively acting on $G(X,Y)$. If there is a 2-fragment in X, then either*

(i) *there is an imprimitive set* $A \subset X$ *with* $|N(A)| - |A| = d(X) - 1$, *or*
(ii) *there is a subset* $A \subseteq X$, *where* $A = X$ *or* A *is an imprimitive set under the action of* Γ *with* $|A| > 2$, *such that the quotient group* $\Gamma_A/(\bigcap_{a\in A}\Gamma_a)$ *is isomorphic to a subgroup of the dihedral group* $D_{|A|}$, *where* $\Gamma_A = \{\sigma \in \Gamma\colon\ \sigma(A) = A\}$.

Proof. By definition we have that for any $x, y \in X$, $\{x, y\}$ is a 2-fragment if and only if $|N(x)\cap N(y)| = d(X) - 1$. We now define a simple graph $H = H(X)$, whose vertex set $V(H)$ is X, and whose edge set $E(H)$ consists of all pairs xy such that $\{x, y\}$ is a fragment in X. Then, each element of Γ induces an automorphism of H so that H is vertex-transitive. As usual, the degree of elements of H is denoted by $d(H)$.

Let H' be a connected component of H and let A be the vertex set of H'. Then $|A| \geqslant 2$. If $|A| = 2$, then A is clearly an imprimitive set in X with $|N(A)| - |A| = d(X) - 1$. Suppose $|A| > 2$ and let xyz be a path in H' for distinct $x, y, z \in A$. Set $N(x) = B \cup \{a\}$ and $N(y) = B \cup \{b\}$, where $B = N(x) \cap N(y)$. Since $yz \in E(H')$, $N(z) = (N(y) \setminus \{c\}) \cup \{d\}$ for some $c \in N(y)$ and $d \in N(z) \setminus N(y)$. If $d \in N(x) \cup N(y)$, then $N(x, y, z) = N(x, y)$, contradicting that $\{x, y\}$ is a fragment. Therefore, $d \notin N(x) \cup N(y)$. So

$$N(x) \cap N(z) = \begin{cases} B & \text{if } c = b, \\ B \setminus \{c\} & \text{if } c \neq b. \end{cases}$$

From this, the following is immediate.

Claim. $N(y) \subset N(x) \cup N(z)$ *if the induced subgraph* $H'[\{x, y, z\}]$ *is a path, and* $|N(x) \cap N(y) \cap N(z)| = d(X) - 1$ *if* $H'[\{x, y, z\}]$ *is a cycle.*

If H' is a complete graph, then from the above claim it follows that $|\bigcap_{x\in A} N(x)| = d(X) - 1$, so $|N(A)| - |A| = d(X) - 1$. If $d(X) \geqslant 2$, we have $|A| < |X|$, hence A is an imprimitive set in X with $|N(A)| - |A| = d(X) - 1$.

If H' is not complete, then there are more than three elements of A, say $x_1, x_2, \ldots, x_m$, such that the induced subgraph $H'[\{x_1, \ldots, x_m\}]$ is a cycle, written as $x_1x_2\cdots x_mx_1$. By definition we see that $|N(x_1, x_2, \ldots, x_s)| \leqslant d(X) - 1 + s$ for $1 \leqslant s \leqslant m - 1$, and equality holds if $N(x_1, x_2, \ldots, x_s) \neq Y$, that is, $\{x_1, x_2, \ldots, x_s\}$ is a fragment. Now, if $N(x_1, x_2, \ldots, x_{m-1}) \neq Y$, then, by the above claim, $N(x_1, x_2, \ldots, x_m) = N(x_1, x_2, \ldots, x_{m-1}) \neq Y$, which yields a contradiction since $|N(x_1, x_2, \ldots, x_m)| - m < d(X) - 1$. Therefore, $N(x_1, x_2, \ldots, x_{m-1}) = Y$. Assume that t is the least index such that $N(x_1, x_2, \ldots, x_t) = Y$, where $2 < t \leqslant m - 1$. This means that every path of length less than t on this cycle is a fragment. In this case, if $d(H) > 2$, then there is an $x \in A\backslash\{x_1, \ldots, x_m\}$ such that xx_{t-1} is an edge of H'. Setting $a \in N(x)\backslash N(x_{t-1})$, we have that $a \in N(x_i)$ for some $i \in [t]\backslash\{t - 1\}$. Then $N(x_i, \ldots, x_{t-1}) = N(x_i, \ldots, x_{t-1}, x)$ if $i < t - 1$, or $N(x_{t-1}, x_t) = N(x_{t-1}, x_t, x)$ if $i = t$. Both cases contradict that $\{x_i, \ldots, x_{t-1}\}$ and $\{x_{t-1}, x_t\}$ are fragments. This proves that H' is a cycle, and hence (ii) holds. □

Proposition 2.4. *Let* $G(X, Y)$ *be as in Proposition* 2.3. *If each fragment of* $G(X, Y)$ *is primitive and there are no* 2*-fragments in* $\mathcal{F}(X, Y)$, *then every nontrivial fragment* $A \in \mathcal{F}(X)$ (*if it exists*) *is balanced, and for each* $a \in A$, *there is a unique nontrivial fragment* B *such that* $A \cap B = \{a\}$.

Proof. Let A be a minimum-sized nontrivial fragment in $\mathcal{F}(X, Y)$. Then, $\phi(A)$ is a maximum-sized fragment in X and Y. Without loss of generality, suppose that $A \in \mathcal{F}(X)$. Then $\phi(A) = Y\backslash N(A)$ and $|Y\backslash N(A)| \geqslant |A|$. We now prove that equality holds, i.e., A is balanced.

Suppose, to the contrary, that $|Y\backslash N(A)| > |A|$. Set $\mathcal{A} = \{\sigma(A)\colon\ \sigma \in \Gamma\}$. As we have mentioned, A is semi-imprimitive, so $|B \cap C| = 1$ or 0 for all distinct $B, C \in \mathcal{A}$. We now define a graph $H = H(\mathcal{A})$, whose vertex set is $\mathcal{A}$, and whose edge set consists of all pairs BC such that $|B \cap C| = 1$ for $A, B \in \mathcal{A}$. Clearly, H is vertex-transitive. Since A is primitive, H is not an empty graph. Suppose that $A \cap B = \{b\}$ for some $B \in \mathcal{A}$ and $b \in A$. Then, for each $a \in A$, the part-transitivity of $G(X, Y)$ implies that there is a $\sigma \in \Gamma$ with $\sigma(b) = a$, hence $\sigma(A) \cap \sigma(B) = \{a\}$. Since $\sigma(A) \neq \sigma(B)$, there is at most one of them equal to A. Suppose $\sigma(A) \neq A$. Then, the semi-imprimitivity of A implies that $\sigma(A) \cap A = \{a\}$.

From this it follows that the degree of vertices in H, denoted $d(H)$, is at least $|A|>2$. Hence H contains a cycle. Let $\mathcal{C}=AA_1\dots A_sA$ be a cycle of minimum length, where $s\geqslant 3$. Then the induced subgraph $H[\{A,A_1,\dots,A_i\}]$ is a path from A to A_i for $i=1,2,\dots,s-1$. By Lemma 2.1, if $N(A\cup A_1\cup\dots\cup A_i)\neq Y$, then both $A\cup A_1\cup\dots\cup A_i$ and $Y\backslash N(A\cup A_1\cup\dots\cup A_i)$ are fragments. Furthermore, if $|Y|-|N(A\cup A_1\cup\dots\cup A_i)|>1$, then the minimality of A implies $|Y|-|N(A\cup A_1\cup\dots\cup A_i)|\geqslant|A|$, hence

$$\begin{aligned}&\bigl|N(A\cup A_1\cup\dots\cup A_{i+1})\bigr|\\&\quad\leqslant\bigl|N(A\cup A_1\cup\dots\cup A_i)\bigr|+\bigl|N(A_{i+1})\bigr|-\bigl|N\bigl((A\cup A_1\cup\dots\cup A_i)\cap A_{i+1}\bigr)\bigr|\\&\quad=\bigl|N(A\cup A_1\cup\dots\cup A_i)\bigr|+|A|-1\leqslant|Y|-1,\end{aligned}$$

i.e., $A\cup A_1\cup\dots\cup A_{i+1}$ is also a fragment. Now, if $|Y|-|N(A\cup A_1\cup\dots\cup A_{s-1})|>1$, then, by Lemma 2.1, $A_s\cap(A\cup\dots\cup A_{s-1})$ is a fragment. However, it is clear that $|A_s\cap(A\cup\dots\cup A_{s-1})|=2$, yielding a contradiction. Therefore, there is a unique index k with $2\leqslant k\leqslant s-1$ such that $|Y|-|N(A\cup A_1\cup\dots\cup A_k)|=1$. That is, $A\cup A_1\cup\dots\cup A_k$ is a maximum-sized fragment, and the induced subgraph $H[A\cup A_1\cup\dots\cup A_k]$ is a path. Set $A'=A\cup A_1\cup\dots\cup A_{k-1}$. Then, it is clear that $|Y|-|N(A')|=|A|$. Note that $k\geqslant 2$. The minimality of $\mathcal{C}$ implies that $(N_H(A)\cap N_H(A_{k-1}))\setminus\{A_1,A_{k-2}\}=\emptyset$. Then for every $B\in(N_H(A)\cup N_H(A_{k-1}))\setminus\{A_1,A_{k-2}\}$, $A'\cup B$ is a maximum-sized fragment. We thus obtain at least $2(d(H)-1)$ maximum-sized fragments in X containing A'. On the other hand, for every maximum-sized fragment $C\in\mathcal{F}(X)$ containing A', we have that $C=X\backslash N(b)$ for some $b\in Y\backslash N(A')$ since $|Y\backslash N(C)|=1$, hence there are at most $|Y\backslash N(A')|=|A|$ maximum-sized fragments in $\mathcal{F}(X)$ containing A', yielding a contradiction because $2(d(H)-1)\geqslant 2(|A|-1)>|A|$. This proves that $|Y\backslash N(A)|=|A|$, i.e., A is balanced.

Now, assume that $A=\{a_1,\dots,a_d\}$, where $d>2$. As we have seen, for each a_i, there is a $\sigma_i\in\Gamma$ such that $A\cap\sigma_i(A)=\{a_i\}$. Thus $A\cup\sigma_i(A)$ is a maximum-sized fragment containing A. The semi-imprimitivity and $d>2$ imply $A\cup\sigma_i(A)\neq A\cup\sigma_j(A)$ if $i\neq j$. Therefore, $A\cup\sigma_i(A)$, $i=1,\dots,d$, are all the maximum-sized fragments containing A. This proves that for every $a\in A$, there is only one nontrivial fragment B with $A\cap B=\{a\}$. $\square$

3. Proofs of Theorem 1.2

With the assumptions in the theorem, we put $\mathcal{X}=\binom{[n]}{a}$ and $\mathcal{Y}=\binom{[n]}{b}$. A bipartite graph $G_1(\mathcal{X},\mathcal{Y})$ is defined by the cross-t-intersecting relation: for $A\in\mathcal{X}$ and $B\in\mathcal{Y}$, $AB\in E(G_1)$ if and only if $|A\cap B|<t$, that is, A and B are not t-intersecting.

For the graph $G_1(\mathcal{X},\mathcal{Y})$, if $a+b\geqslant t+n$, there is no edge in $G_1(\mathcal{X},\mathcal{Y})$, and if $(n,t)=(a+b,1)$, $G_1(\mathcal{X},\mathcal{Y})$ is a union of disjoint edges. In both cases, Theorem 1.2 is trivial. Therefore, we always assume that $a+b<n+t$ and $(n,t)\neq(a+b,1)$ in $G_1(\mathcal{X},\mathcal{Y})$. In this case, it is easy to see that $G_1(\mathcal{X},\mathcal{Y})$ is connected and non-complete since $t<\min\{a,b\}$.

Clearly, S_n acts transitively on $\mathcal{X}$ and $\mathcal{Y}$, respectively, in a natural way, and preserves the cross-t-intersecting. Therefore, $d(\mathcal{X})=|N_{G_1}(A)|$ for each $A\in\mathcal{X}$, and $d(\mathcal{Y})=|N_{G_1}(B)|$ for each $B\in\mathcal{Y}$. It is easy to see that for each $A\in\mathcal{X}$,

$$N_{G_1}(A)=\left\{B\in\binom{[n]}{b}:\ |A\cap B|<t\right\}=\bigcup_{0\leqslant i\leqslant t-1}\left\{B\in\binom{[n]}{b}:\ |A\cap B|=i\right\},$$

hence $|N_{G_1}(A)|=\sum_{i=0}^{t-1}\binom{a}{i}\binom{n-a}{b-i}$. Similarly, we have $|N_{G_1}(B)|=\sum_{i=0}^{t-1}\binom{b}{i}\binom{n-b}{a-i}$ for each $B\in\mathcal{Y}$.

It is well known that for each $A\in\binom{[n]}{k}$, the stabilizer of A is a maximal subgroup of S_n subject to $n\neq 2k$ [25]. Therefore, the action of S_n on $\binom{[n]}{k}$ is imprimitive if and only if $n=2k\geqslant 4$, and the only imprimitive sets are all pairs of complementary subsets. Now, we verify that each pair $\{A,\overline{A}\}$ is not a fragment of $G_1(\mathcal{X},\mathcal{Y})$, where $A\in\mathcal{X}=\binom{[n]}{a}$ with $n=2a\geqslant 4$ and $\overline{A}=[n]\setminus A$.

Note that $(n,t)\neq(a+b,1)$ and $b<a+t$. For every pair A and $\overline{A}$ in $\binom{[n]}{a}$, it is easy to verify that $\{C\cup\{i\}:\ C\in\binom{\overline{A}}{a-1},\ i\in A\}\subseteq N_{G_1}(A)\backslash N_{G_1}(\overline{A})$ if $b=a$ and $t>1$; $\binom{\overline{A}}{b}\subseteq N_{G_1}(A)\backslash N_{G_1}(\overline{A})$ if $b<a$; and

$\{\overline{A} \cup C\colon\ C \in \binom{A}{b-a}\} \subseteq N_{G_1}(A) \backslash N_{G_1}(\overline{A})$ if $b > a$. This implies $|N_{G_1}(A) \backslash N_{G_1}(\overline{A})| > 1$, hence $|N_{G_1}(A) \cup N_{G_1}(\overline{A})| > |N_{G_1}(A)| + 1$. We thus prove that $\{A, \overline{A}\}$ is not a fragment of $G_1(\mathcal{X}, \mathcal{Y})$ for every $A \in \binom{[n]}{a}$. Similarly, we have the same result for $n = 2b$ and $\mathcal{Y} = \binom{[n]}{b}$. We thus obtain that every fragment in $\mathcal{X}$ and $\mathcal{Y}$ is primitive. Then, by Theorem 1.1, inequality (1.3) holds.

To complete the proof of Theorem 1.2 we need to characterize all nontrivial fragments. Suppose there is a nontrivial fragment of $G_1(\mathcal{X}, \mathcal{Y})$. Without loss of generality we assume that $\mathcal{S}$ is a minimal fragment in $\binom{[n]}{a}$. By Theorem 1.1, $\binom{n}{a} = \binom{n}{b}$, i.e., $b = a$ or $b = n - a$. Clearly, S_n is not isomorphic to a subgroup of $D_{|\mathcal{X}|}$ for $n \geqslant 4$. Therefore, by Proposition 2.3, there are no 2-fragments of $G_1(\mathcal{X}, \mathcal{Y})$, which implies that $\mathcal{S}$ is balanced.

For each $C \subseteq [n]$, S_C is embedded within S_n in a natural way: for $\sigma \in S_C$, let σ fix the elements of $\bar{C}$. Now, take a $C \in \mathcal{S}$ and let $\Gamma = S_C \times S_{\overline{C}}$ and $\Gamma_{\mathcal{S}} = \{\sigma \in \Gamma\colon\ \sigma(\mathcal{S}) = \mathcal{S}\}$. Then $C \in \sigma(\mathcal{S})$ for each $\sigma \in \Gamma_{\mathcal{S}}$. Since $\mathcal{S}$ has more than one element, we have $\Gamma_{\mathcal{S}} \neq \Gamma$. Otherwise, $S_C \times S_{\overline{C}}$ and $S_B \times S_{\overline{B}}$ (for some $B \in \mathcal{S} \backslash \{C\}$) will generate all of S_n so that $\mathcal{S} = \binom{[n]}{a}$, yielding a contradiction. Then, by Proposition 2.4 we have that $[\Gamma : \Gamma_{\mathcal{S}}]$, the index of $\Gamma_{\mathcal{S}}$ in Γ, equals 2. Now, let $\Gamma_{\mathcal{S}}[C]$ be the projection of $\Gamma_{\mathcal{S}}$ onto S_C. Then, $\Gamma_{\mathcal{S}}[C]$ is a subgroup of S_C of index $\leqslant 2$. That is, $\Gamma_{\mathcal{S}}[C] = S_C$ or A_C. From this we see that $A_C \times S_{\overline{C}}$ and $S_C \times A_{\overline{C}}$ are the only index-2 subgroups of Γ. That is, $\Gamma_{\mathcal{S}} = A_C \times S_{\overline{C}}$ or $S_C \times A_{\overline{C}}$. Clearly, $a = |B \cap C| + |B \cap \overline{C}|$ for each $B \in \mathcal{S} \backslash \{C\}$. If $|B \cap C| > 1$, let $(i\ j)$ be a transposition, where $i, j \in B \cap C$. Then, $(i\ j)$ fixes both C and B. The semi-imprimitivity of $\mathcal{S}$ implies $(i, j) \in \Gamma_{\mathcal{S}}$. This yields $\Gamma_{\mathcal{S}} = S_C \times A_{\overline{C}}$. From this process it follows that, for each $B \in \mathcal{S}$, there exists at most one of $|B \cap C|$ and $|B \cap \overline{C}|$ greater than 1. Note that if $B \subseteq \overline{C}$, then S_C and S_B fix both C and B, i.e., $S_C \times S_B \subseteq \Gamma_{\mathcal{S}}$. It is clear, however, that neither $A_C \times S_{\overline{C}}$ nor $S_C \times A_{\overline{C}}$ contain $S_C \times S_B$. We therefore obtain that $|C \cap B| = 1$ for every $B \in \mathcal{S}$, or $|C \cap B| = a - 1$ for every $B \in \mathcal{S}$. We now determine all the nontrivial fragments of $G_1(\mathcal{X}, \mathcal{Y})$.

Suppose $|C \cap B| = 1$ for every $B \in \mathcal{S}$. Without loss of generality we assume $C \cap B = \{1\}$ for some $B \in \mathcal{S}$. In this case, if $a > 2$, then $|B \cap \overline{C}| \geqslant 2$, so $\Gamma_{\mathcal{S}} = A_C \times S_{\overline{C}}$. On the other hand, we can find distinct $i, j \in C$ such that $(1\ i\ j)(B) = B \backslash \{1\} \cup \{i\} \in \mathcal{S}$ because $(1\ i\ j) \in A_C$. From this it follows that $(1\, i)(\mathcal{S})$ contains more than one element of $\mathcal{S}$, hence $(1\ i) \in \Gamma_{\mathcal{S}}$. The contradiction proves $a = 2$. Thus $\mathcal{S}$ consists of all 2-subsets $\{1, i\}$ for $i \in [2, n]$. Since $t < \min\{a, b\}$ and $(n, t) \neq (a + b, 1)$, we have $t = 1$ and $b = 2$. Then $d_1(\mathcal{X}) = \binom{n-2}{2}$ and $N_{G_1}(\mathcal{S}) = \binom{[2,n]}{2}$, satisfying $|N_{G_1}(\mathcal{S})| - |\mathcal{S}| = d_1(\mathcal{X}) - 1$, that is, $\mathcal{S}$ is a fragment in $\binom{[n]}{2}$ and $\overline{N}_{G_1}(\mathcal{S}) = \mathcal{S} = \{A \in \binom{[n]}{2}\colon\ 1 \in A\}$.

Suppose now $|C \cap B| = a - 1 > 1$ for every $B \in \mathcal{S}$. In this case, we may similarly prove that $n - a = 2$, $b = a$, $t = n - 3$ and $\Gamma_{\mathcal{S}} = S_C$. Thus $\mathcal{S} = \{\sigma(B \cap C) \cup \{i\}\colon\ \sigma \in S_C$ and $\{i\} = B \cap \overline{C}\} \cup \{C\} = \binom{A}{n-2}$ where $A = B \cup C$ is an $(a+1)$-subset of $[n]$. It is easy to verify that $\mathcal{S}$ is a fragment in $\binom{[n]}{n-2}$ and $\overline{N}_{G_1}(\mathcal{S}) = \mathcal{S}$.

4. Proofs of Theorem 1.3

Corresponding cross-Sperner property, we define another bipartite graph $G_2(\mathcal{X}, \mathcal{Y})$: for $A \in \mathcal{X}$ and $B \in \mathcal{Y}$, $AB \in E(G_1)$ if and only if $A \subseteq B$ or $B \subseteq A$.

For the graph $G_2(\mathcal{X}, \mathcal{Y})$, if $a = b$, it is clear that $G_2(\mathcal{X}, \mathcal{Y})$ is a union of disjoint edges, and in this case, Theorem 1.3 is trivial. So we assume $a < b$ in $G_2(\mathcal{X}, \mathcal{Y})$. Then $G_2(\mathcal{X}, \mathcal{Y})$ is connected and non-complete. Moreover, $G_2(\mathcal{X}, \mathcal{Y})$ is part-transitive under the action induced by S_n. Therefore, $d(\mathcal{X}) = |N_{G_2}(A)|$ for each $A \in \mathcal{X}$, and $d(\mathcal{Y}) = |N_{G_2}(B)|$ for each $B \in \mathcal{Y}$. It is easy to see that for each $A \in \mathcal{X}$,

$$N_{G_2}(A) = \left\{B \in \binom{[n]}{b}\colon\ A \subset B\right\},$$

hence $|N_{G_2}(A)| = \binom{n-a}{b-a}$. Similarly, we have $|N_{G_2}(B)| = \binom{b}{a}$ for each $B \in \mathcal{Y}$.

From the proof of Theorem 1.2, we know that the only imprimitive sets of $\mathcal{X}$ are all pairs of complementary subsets subject to $n = 2a$. For each pair $\{A, \overline{A}\}$ of $\mathcal{X}$ with $n = 2a \geqslant 4$, it is easy to see that $N_{G_2}(A) \cap N_{G_2}(\overline{A}) = \emptyset$ holds for every $A \in \binom{[n]}{a}$, so $|N_{G_2}(A) \cup N_{G_2}(\overline{A})| = 2\binom{b}{a} > |N_{G_2}(A)| + 1$. We thus prove that $\{A, \overline{A}\}$ is not a fragment of $G_2(\mathcal{X}, \mathcal{Y})$ for every $A \in \mathcal{X}$. Similarly, we have the same result for $n = 2b$ and $\mathcal{Y}$. We thus obtain that every fragment in $\mathcal{X}$ and $\mathcal{Y}$ is primitive. Then, by Theorem 1.1, inequality (1.4) holds.

To complete the proof of Theorem 1.3, we only need to characterize all nontrivial fragments. Suppose there is a nontrivial fragment of $G_2(\mathcal{X},\mathcal{Y})$. Without loss of generality we assume that $\mathcal{S}$ is a minimal fragment in $\binom{[n]}{a}$. By Theorem 1.1, $\binom{n}{a}=\binom{n}{b}$, i.e., $b=a$ or $b=n-a$. In the similarly way as in the proof of Theorem 1.2, we can also obtain that $\mathcal{S}$ is balanced and $|C\cap B|=1$ for every $B\in\mathcal{S}$, or $|C\cap B|=a-1$ for every $B\in\mathcal{S}$. In the following, we determine all the nontrivial fragments of $G_2(\mathcal{X},\mathcal{Y})$.

Suppose $|C\cap B|=a-1>1$ for every $B\in\mathcal{S}$. Reproducing the proceeding of Theorem 1.2, we may also prove that $n-a=2$ and $b=2$. Note that $b>a$. So we have $n=3$, $a=1$ and $b=2$, contradicting that $a-1>1$. Therefore, $|C\cap B|=1$ for every $B\in\mathcal{S}$. Without loss of generality we assume $C\cap B=\{1\}$ for some $B\in\mathcal{S}$. In this case, if $a>2$, then $|B\cap\overline{C}|\geqslant 2$, so $\Gamma_{\mathcal{S}}=A_C\times S_{\overline{C}}$. On the other hand, we can find distinct $i,j\in C$ such that $(1\ i\ j)(B)=B\backslash\{1\}\cup\{i\}\in\mathcal{S}$ because $(1\ i\ j)\in A_C$. From this it follows that $(1\ i)(\mathcal{S})$ contains more than one element of $\mathcal{S}$, hence $(1\ i)\in\Gamma_{\mathcal{S}}$. The contradiction proves $a=2$ and $b=n-2$. Thus $\mathcal{S}$ consists of all 2-subsets $\{1,i\}$ for $i\in[2,n]$. It is easy to verify that $\mathcal{S}$ is a fragment, and $\overline{N}_{G_2}(\mathcal{S})=\{B\in\binom{[n]}{n-2}\colon\ 1\notin B\}$.

5. Proofs of Theorem 1.4

Put $\mathcal{X}=\begin{bmatrix}V\\a\end{bmatrix}$ and $\mathcal{Y}=\begin{bmatrix}V\\b\end{bmatrix}$. According to the cross-t-intersecting relation between $\mathcal{X}$ and $\mathcal{Y}$, we define the bipartite graph $G_3(\mathcal{X},\mathcal{Y})$ as that: For $A\in\mathcal{X}$ and $B\in\mathcal{Y}$, $AB\in E(G_3)$ if and only if $\dim(A\cap B)<t$. Analogous to the families of sets, $G_3(\mathcal{X},\mathcal{Y})$ is connected and non-complete, subject to the conditions in Theorem 1.4.

Let $GL(V)$ denote the general linear group of V, which consists of all invertible linear transformations of V. Clearly, $GL(V)$ transitively acts on $\mathcal{X}$ and $\mathcal{Y}$, respectively, in a natural way, and preserves the cross-t-intersecting. So $d(\mathcal{X})=|N_{G_3}(A)|$ for $A\in\mathcal{X}$. It is easy to see that

$$N_{G_3}(A)=\bigcup_{i=0}^{t-1}N_i^b(A),$$

where $N_i^b(A)=\{T\in\begin{bmatrix}V\\b\end{bmatrix}\colon\ \dim(T\cap A)=i\}$. To determine $|N_{G_3}(A)|$, we need a useful result, stated as a lemma as follows.

Lemma 5.1. *(See [7, Proposition 2.2].) Let $A\in\begin{bmatrix}V\\a\end{bmatrix}$. Then $|N_0^b(A)|=\begin{bmatrix}n-a\\b\end{bmatrix}q^{ab}$.*

Suppose $i>0$ and let C be an arbitrary i-subspace of A. We consider the quotient space V/C. Then $\dim(V/C)=n-i$, $\dim(A/C)=a-i$ and $|N_0^{b-i}(A/C)|=\begin{bmatrix}n-a\\b-i\end{bmatrix}q^{(a-i)(b-i)}$. So $|N_i^b(A)|=\begin{bmatrix}a\\i\end{bmatrix}|N_0^{b-i}(A/C)|=\begin{bmatrix}a\\i\end{bmatrix}\begin{bmatrix}n-a\\b-i\end{bmatrix}q^{(a-i)(b-i)}$ (see also [29, Lemma 4] and [26, Lemma 2.4]). Thus

$$\left|N_{G_3}(A)\right|=\sum_{i=0}^{t-1}q^{(a-i)(b-i)}\begin{bmatrix}a\\i\end{bmatrix}\begin{bmatrix}n-a\\b-i\end{bmatrix}=d(\mathcal{X}).$$

Similarly, $d(\mathcal{Y})=\sum_{i=0}^{t-1}q^{(a-i)(b-i)}\begin{bmatrix}b\\i\end{bmatrix}\begin{bmatrix}n-b\\a-i\end{bmatrix}$.

For a subspace A of V, we denote the stabilizer of A in $GL(V)$ by $GL(V|A)$. It is well known that for $A\in\mathcal{X}$, $GL(V|A)$ is a maximal subgroup of $GL(V)$ [1], so the action of $GL(V)$ on $\mathcal{X}$ is primitive. Then, by Theorem 1.1, inequality (1.5) and inequality (1.6) hold, and each nontrivial fragment of $G_3(\mathcal{X},\mathcal{Y})$ (if they exist) is a semi-imprimitive set under the action of $GL(V)$.

To complete the proof of Theorem 1.4, we prove that there are no nontrivial fragments of $G_3(\mathcal{X},\mathcal{Y})$. Suppose, to the contrary, that there is a nontrivial fragment of $G_3(\mathcal{X},\mathcal{Y})$. Without loss of generality we assume that $\mathcal{S}$ is a minimal fragment in $\mathcal{X}$. By Theorem 1.1, $\begin{bmatrix}n\\a\end{bmatrix}=\begin{bmatrix}n\\b\end{bmatrix}$, i.e., $b=a$ or $b=n-a$. Clearly, $GL(V)/K$ is not isomorphic to a subgroup of $D_{|\mathcal{X}|}$, where K is the kernel of the action of $GL(V)$ on $\begin{bmatrix}V\\a\end{bmatrix}$ or $\begin{bmatrix}V\\b\end{bmatrix}$. Therefore, by Proposition 2.3, there are no 2-fragments in $\mathcal{F}(\mathcal{X},\mathcal{Y})$, which implies that $\mathcal{S}$ is balanced.

Take $C\in\mathcal{S}$, write $\Gamma=GL(V|C)$ and $\Gamma_{\mathcal{S}}=\{\sigma\in\Gamma\colon\ \sigma(\mathcal{S})=\mathcal{S}\}$. Then, $\Gamma\neq\Gamma_{\mathcal{S}}$, and again by Proposition 2.4, $[\Gamma:\Gamma_{\mathcal{S}}]=2$ so that $\Gamma=\Gamma_{\mathcal{S}}\cup\gamma\Gamma_{\mathcal{S}}$ for some $\gamma\in\Gamma$. Thus $\mathcal{S}$ and $\gamma(\mathcal{S})$ are the only nontrivial fragments containing C. From the structure of Γ it follows that Γ is transitive on $N_i^a(C)$ for each $i = 0, 1, \dots, a$, whenever $N_i^a(C) \neq \emptyset$. Set $\mathcal{S}_i = \mathcal{S} \cap N_i^a(C)$ for $0 \leqslant i < a$. If $\mathcal{S}_i \neq \emptyset$, then

$$N_i^a(C) = \mathcal{S}_i \cup \gamma(\mathcal{S}_i) = \left\{ L + R \colon L \in \begin{bmatrix} C \\ i \end{bmatrix} \text{ and } R \in N_0^{a-i}(C) \right\}.$$

From this we see that $\Gamma_{\mathcal{S}}$ is transitive on $\mathcal{S}_i$. It is clear that the restriction of Γ on C is $GL(C)$. Therefore, the induced action of Γ on $\left[\begin{smallmatrix} C \\ i \end{smallmatrix}\right]$ is primitive, thus the action of $\Gamma_{\mathcal{S}}$ on $\left[\begin{smallmatrix} C \\ i \end{smallmatrix}\right]$ is transitive. This means that if $L_0 + R_0 \in \mathcal{S}_i$ for some $R_0 \in N_0^{a-i}(C)$, then $L + R_0 \in \mathcal{S}_i$ for every $L \in \left[\begin{smallmatrix} C \\ i \end{smallmatrix}\right]$. We complete the proof by considering of two cases.

Case 1: $n - a = a - i$. Suppose that $L_0 + R_0 \in \mathcal{S}_i$ and $\{\alpha_1, \dots, \alpha_{a-i}\}$ is a basis of R_0. Then bases of elements of $N_0^{a-i}(C)$ are of the form $\{\alpha_1 + \beta_1, \dots, \alpha_{a-i} + \beta_{a-i}\}$, where $\beta_i \in C$. Put $Q = \{R \in N_0^{a-i}(C)\colon L + R \in \mathcal{S}_i$ for some $L \in \left[\begin{smallmatrix} C \\ i \end{smallmatrix}\right]\}$. Then $R_0 \in Q$. For given $\beta_1, \dots, \beta_{a-i} \in C$, let R_j be the subspace generated by $\alpha_1 + \beta_1, \dots, \alpha_j + \beta_j, \alpha_{j+1}, \dots, \alpha_{a-i}$. Assume $R_j \in Q$. Then the above discussion implies $L + R_j \in Q$ for every $L \in \left[\begin{smallmatrix} C \\ i \end{smallmatrix}\right]$. Thus, we can take an $L \in \left[\begin{smallmatrix} C \\ i \end{smallmatrix}\right]$ containing β_{j+1} so that $L + R_j = L + R_{j+1}$, that is, $R_{j+1} \in Q$. This proves $\mathcal{S}_i = N_i^a(C)$, yielding a contradiction.

Case 2: $n - a > a - i$. Consider the natural map ν from V onto the quotient space V/C, that is, $\nu(A) = (A + C)/C$, written as $\bar{A}$, for any subspace A of V. Then $\nu(N_i^{a-i}(C)) = \left[\begin{smallmatrix} V/C \\ a-i \end{smallmatrix}\right]$. It is clear that Γ acts on V/C and Γ/K is isomorphic to $GL(V/C)$, where K is the kernel of the action. Then the primitivity of the action implies that $\Gamma_{\mathcal{S}} K/K$ is transitive on $\left[\begin{smallmatrix} V/C \\ a-i \end{smallmatrix}\right]$. This means that for each $\bar{R}_0 \in \left[\begin{smallmatrix} V/C \\ a-i \end{smallmatrix}\right]$, there is an $R_0 \in N_0^{a-i}(C)$ such that $\nu(R_0) = \bar{R}_0$ and $L_0 + R_0 \in \mathcal{S}_i$ for some $L_0 \in \left[\begin{smallmatrix} C \\ i \end{smallmatrix}\right]$. Then, by Case 1 we prove $\mathcal{S}_i = N_i^a(C)$, also yielding a contradiction.

We thus proved that there are no nontrivial fragments of $G_3(\mathcal{X}, \mathcal{Y})$.

6. Proof of Theorem 1.5

Set $\mathcal{X} = \left[\begin{smallmatrix} V \\ a \end{smallmatrix}\right]$ and $\mathcal{Y} = \left[\begin{smallmatrix} V \\ b \end{smallmatrix}\right]$. According to the cross-Sperner relation between $\mathcal{X}$ and $\mathcal{Y}$, we define the bipartite graph $G_4(\mathcal{X}, \mathcal{Y})$ as that: For $A \in \mathcal{X}$ and $B \in \mathcal{Y}$, $AB \in E(G_2)$ if and only if $A \subseteq B$ or $B \subseteq A$. It is easy to see that $G_4(\mathcal{X}, \mathcal{Y})$ is connected and non-complete, subject to the conditions in Theorem 1.5. Moreover, $G_4(\mathcal{X}, \mathcal{Y})$ is part-transitive under the action induced by $GL(V)$. So $d(\mathcal{X}) = |N_{G_4}(A)|$ for $A \in \mathcal{X}$. It is easy to see that

$$N_{G_4}(A) = \left\{ B \in \begin{bmatrix} V \\ b \end{bmatrix} \colon A \subset B \right\} \quad \text{for } A \in \begin{bmatrix} V \\ a \end{bmatrix},$$

and

$$N_{G_4}(B) = \left\{ A \in \begin{bmatrix} V \\ a \end{bmatrix} \colon A \subset B \right\} \quad \text{for } B \in \begin{bmatrix} V \\ b \end{bmatrix},$$

and by Lemma 5.1, we have $d(\mathcal{X}) = \left[\begin{smallmatrix} n-a \\ b-a \end{smallmatrix}\right]$ and $d(\mathcal{Y}) = \left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right]$.

Note that the action of $GL(V)$ on $\mathcal{X}$ and $\mathcal{Y}$ are both primitive. Then every fragments in $\mathcal{X}$ or $\mathcal{Y}$ is primitive, and hence inequality (1.6) holds by Theorem 1.1. Reproducing the proceeding of Theorem 1.4, we can also prove that $G_4(\mathcal{X}, \mathcal{Y})$ has neither 2-fragments nor balanced fragments. Therefore, there are no nontrivial fragments of $G_4(\mathcal{X}, \mathcal{Y})$.

7. Proof of Theorem 1.6

We first prove a general result. Let Γ be a transitive permutation group on Ω with the identity 1. By the group and a positive integer t with $1 \leqslant t \leqslant |\Omega| - 2$ we define a simple graph, written as $G_t = G_t(\Gamma)$, whose vertex set is Γ, and whose edge set consists of all pairs $\sigma\tau$ such that $|\{x \in \Omega\colon \sigma(x) = \tau(x)\}| < t$. Let Γ_L and Γ_R denote the left and right regular action on Γ, respectively. Then $\Gamma_L \times \Gamma_R$ (not necessarily a direct product) induces an automorphism group of $G_t(\Gamma)$. In a natural way, we can define the bipartite graph $G_t(\Gamma, \Gamma)$, which is part-transitive under the action of Γ_L and Γ_R.

Lemma 7.1. *Suppose that A is an imprimitive set in Γ under the action of $\Gamma_L \times \Gamma_R$. Then A is a coset of a nontrivial normal subgroup of Γ.*

Proof. Since A is an imprimitive set, we have that $1 < |A| < |\Gamma|$, and for every $\alpha \in \Gamma$, αA is also an imprimitive set. Without loss of generality we assume that $1_\Gamma \in A$, where 1_Γ is the identity of Γ. From this it follows that $\alpha \in \alpha A$ and $1_\Gamma \in \alpha^{-1} A$ for each $\alpha \in A$, hence $\alpha A = \alpha^{-1} A = A$, which implies that A is a subgroup of Γ. Furthermore, for every $\gamma \in \Gamma$, $1_\Gamma \in (\gamma^{-1} A \gamma) \cap A$, hence $\gamma^{-1} A \gamma = A$, proving that A is a normal subgroup of Γ. $\square$

We now consider the graph $G_t(S_n, S_n)$ where $n \geqslant 4$ and $1 \leqslant t \leqslant n-2$. For $0 \leqslant i < n$, by $\mathcal{D}_n^i$ we denote the set of all permutations in S_n which have exactly i fixed points. The elements of $\mathcal{D}_n^0$ are the derangements of $[n]$. As usual, set $|\mathcal{D}_n^0| = d_n$. By definition, $G_t(S_n, S_n)$ is the Cayley graph on S_n generated by $\mathcal{G}_t$, where $\mathcal{G}_t = \bigcup_{i=0}^{t-1} \mathcal{D}_n^i$ (cf. [21]). It is not difficult to compute that for every $\sigma \in S_n$,

$$\left|N(\sigma)\right| = |\mathcal{G}_t \sigma| = |\mathcal{G}_t| = \sum_{i=0}^{t-1} \binom{n}{i} d_{n-i}.$$

Let $\mathcal{S}$ be a fragment of $G_t(S_n, S_n)$. Then for any $\sigma \in S_n$, $\sigma\mathcal{S}$ is also a fragment. Without loss of generality, we assume that $1_{S_n} \in \mathcal{S}$. If $\mathcal{S}$ is imprimitive, then Lemma 7.1 implies that $\mathcal{S}$ is a nontrivial normal subgroup of S_n. It is well known that the only nontrivial normal subgroups of S_n are A_n and the quaternary group $V_4 = \{1_{S_n}, (12)(34), (13)(24), (14)(23)\}$ for $n = 4$. Since A_n has index 2 in S_n and $\mathcal{G}_t \not\subset A_n$, $\mathcal{G}_t A_n = S_n$, hence A_n is not a fragment in S_n for $n \geqslant 4$. And, for $n = 4$ and $t = 1, 2$, it is straightforward to verify that V_4 is not a fragment in S_4. We thus prove that every fragment in S_n is primitive. Then, by Theorem 1.1 we obtain inequality (1.7). Moreover, by Proposition 2.3, it is easy to verify that $G_t(S_n, S_n)$ has no 2-fragments. We now prove that $G_t(S_n, S_n)$ has only trivial fragments.

Suppose that $G_t(S_n, S_n)$ has a nontrivial fragment $\mathcal{S}$. Then, by Proposition 2.4, $\mathcal{S}$ is balanced and $|\mathcal{S}| > 2$. Without loss of generality we may assume $1_{S_n} \in \mathcal{S}$. Set $H = \{h \in S_n\colon\ h\mathcal{S} = \mathcal{S}\}$. Clearly, H is a subgroup of S_n. If $H = \{1_{S_n}\}$, then $\sigma\mathcal{S} = \tau\mathcal{S}$ implies $\sigma = \tau$ for any $\sigma, \tau \in S_n$, hence for any distinct $a, b \in \mathcal{S}$, by the semi-imprimitivity of $\mathcal{S}$, we have $a^{-1}\mathcal{S} \cap b^{-1}\mathcal{S} = \{1_{S_n}\}$. We thus obtain more than two $|\mathcal{S}|$-fragments containing 1_{S_n}, contradicting Proposition 2.4. Therefore, $|H| > 1$ and $\mathcal{S} = \bigcup_{b \in S} Hb$. For each $a \in \mathcal{S}$, it is evident that $Ha \subset \mathcal{S} \cap \mathcal{S}a$. So the semi-imprimitivity of $\mathcal{S}$ implies that $\mathcal{S} = \mathcal{S}a$, which implies that $\mathcal{S}$ is a subgroup of S_n. We have seen that $\mathcal{S}$ is not normal i.e., there is a $\sigma \in S_n$ with $\sigma^{-1}\mathcal{S}\sigma \neq \mathcal{S}$. However, each $\sigma^{-1}\mathcal{S}\sigma$ contains 1_{S_n}. Again by Proposition 2.4, the normalizer $\mathcal{N}_{S_n}(\mathcal{S})$ is an index-2 subgroup of S_n, i.e., $\mathcal{N}_{S_n}(\mathcal{S}) = A_n$ because A_n is the only index-2 subgroup of S_n. So $\mathcal{S}$ is a normal subgroup of A_n. It is well known that A_n is a simple group for $n \geqslant 5$, therefore A_n has no nontrivial normal subgroup for $n \geqslant 5$, and A_4 has the only nontrivial normal subgroup V_4. We have seen that neither A_n nor V_4 are fragments of $G_t(S_n, S_n)$. We thus prove that the graph has no nontrivial fragments. This completes the proof.

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